Lower bound for energies of harmonic tangent unit-vector fields on convex polyhedra

A Majumdar[†,‡], JM Robbins[†] & M Zyskin[†*]
[†]School of Mathematics
University of Bristol, University Walk, Bristol BS8 1TW, UK
[‡] Hewlett-Packard Laboratories
Filton Road, Stoke Gifford, Bristol BS12 6QZ, UK

8 June, 2004.

Abstract. We derive a lower bound for energies of harmonic maps of convex polyhedra in $\mathbb{R}^3$ to the unit sphere S^2, with tangent boundary conditions on the faces. We also establish that C^∞ maps, satisfying tangent boundary conditions, are dense with respect to the Sobolev norm, in the space of continuous tangent maps of finite energy.
Mathematics Subject Classifications (2000). 58E20, 35J55.
Key Words. Bi-stable nematic devices, liquid crystals, harmonic maps, polyhedra, tangent boundary conditions, lower bounds.

*m.zyskin@bristol.ac.uk

"The unburdened mind is joy supreme"

Gel-sang-gya-tso, 1708-1757.

1 Introduction

We consider maps $\mathbf{n}$ from a convex polyhedron $P \subset \mathbb{R}^3$ to the unit sphere S^2, which we regard as unit-vector fields on P. $\mathbf{n}$ is said to satisfy *tangent boundary conditions*, or to be *tangent*, if, on the faces of P, $\mathbf{n}$ is tangent to the faces. This implies that, on the edges of P, $\mathbf{n}$ is parallel to the edges, and, therefore, discontinuous at the vertices. We consider maps which are continuous away from vertices and which belong to the Sobolev space $W^{1,2}(P)$. There are homotopically inequivalent classes of such maps [2]. In this paper, we derive a lower bound for the energy

$$E[\mathbf{n}] = \int\int\int_P (\nabla\mathbf{n})^2 \, dV \tag{1}$$

for each homotopy class. We also establish that C^∞ tangent unit-vector fields are dense in the space of continuous tangent unit-vector fields in $W^{1,2}(P)$ with respect to the Sobolev norm.

This work is part of a study of liquid crystals in polyhedral geometries started in [2]. We have been motivated by applications to the design of bi-stable liquid crystal displays (see, eg, [3]) as well as by mathematical considerations.

A nematic liquid crystal is a suspension of rod-shaped molecules in a liquid substrate. The molecules have a preferred average orientation at every point in space. This preferred orientation is described by a *director field* – a unit-vector field with opposite orientations identified [1]. We are only considering continuous director fields in a simply connected domain, in which case an orientation can be chosen arbitrarily at one point and defined elsewhere by continuity, thus yielding a unit-vector field $\mathbf{n}$. (Quasi-) stable configurations are (local) minima of a certain energy functional, the Frank energy [1]

$$\begin{aligned} E_F[\mathbf{n}] = \int\int\int_V \big(& K_1(\operatorname{div}\mathbf{n})^2 + K_2(\mathbf{n}\cdot\operatorname{curl}\mathbf{n})^2 + K_3(\mathbf{n}\times\operatorname{curl}\mathbf{n})^2 + \\ & + K_4 \operatorname{div}\left[(\mathbf{n}\cdot\nabla)\mathbf{n} - \mathbf{n}\operatorname{div}\mathbf{n}\right]\big)\, dV. \end{aligned} \tag{2}$$

In the 'one-constant approximation' $K_1 = K_2 = K_3 = K_4 = 1$, and the Frank energy reduces to (1).

Solutions of the Euler-Lagrange equation corresponding to (1), subject to appropriate boundary conditions, are harmonic maps of P to S^2. The boundary conditions are determined by the substrate and surface treatment used. In the cases being considered here, it is strongly energetically favorable for the vector field to be tangent to the boundary.

Let $\bar{P} \subset \mathbb{R}^3$ be a convex polyhedron. Let $\mathbf{v}^a$, $a = 1, \ldots, v$, label the vertices, E^b, $b = 1, \ldots, e$, the edges and F^c, $c = 1, \ldots, f$, the faces of P respectively. $\mathbf{F}^c$ is the outward

unit-normal vector to F^c. Let P denote $\bar{P}$ without its vertices. Let $C^0_{\mathfrak{T}}(P)$ denote the space of continuous tangent unit-vector fields on P. We shall also have occasion to refer to $C^k_{\mathfrak{T}}(P)$, the space of C^k tangent unit-vector fields, and $C^\infty_{\mathfrak{T}}(P)$, the space of smooth tangent unit-vector fields.

We say that $\mathbf{n}$, $\mathbf{n}' \in C^0_{\mathfrak{T}}(P)$ are homotopic, $\mathbf{n} \sim \mathbf{n}'$, if there exists a continuous map $\mathbf{H} : P \times [0,1] \to S^2; (\mathbf{x}, t) \mapsto \mathbf{H}_t(\mathbf{x})$, such that $\mathbf{H}_t \in C^0_{\mathfrak{T}}(P)$ for $t \in [0,1]$ and $\mathbf{H}_0 = \mathbf{n}$, $\mathbf{H}_1 = \mathbf{n}'$.

It is shown in [2] that homotopy classes of tangent unit-vector fields are classified by a set of invariants, which we call *edge orientations*, *kink numbers* and *wrapping numbers*. These are defined as follows:

Edge orientations: The edge orientation $\mathbf{e}^b(\mathbf{n})$ is the value of $\mathbf{n}$ on the edge E^b.

Kink numbers: Let γ^{ac} denote a path on F^c, positively oriented with respect to $\mathbf{F}^c$, between the pair of edges with common vertex $\mathbf{v}^a$. The image of γ^{ac} under $\mathbf{n}$ describes an arc on C^c, the great circle in S^2 parallel to F^c. The kink number $k^{ac}(\mathbf{n})$ is the degree of the closed path on C^c obtained by closing $\mathbf{n}(\gamma^{ac})$ with the shortest arc between its endpoints. Since $\mathbf{n}$ is continuous away from vertices, its restriction to any closed path on F^c away from vertices has degree zero. This implies the following sum rule for the kink numbers: Let $q^c(\mathbf{n})$ denote the number of vertices of F^c at which the edge orientations are oppositely oriented with respect to the normal $\mathbf{F}^c$. Then $\sum_{\mathbf{v}^a \in F^c} k^{ac}(\mathbf{n}) = 1 - \frac{1}{2} q^c(\mathbf{n})$.

Wrapping numbers. Choose $\mathbf{s} \in S^2$ such that $\mathbf{s}$ is not tangent to any of the faces of P. For each vertex $\mathbf{v}^a$, choose an outward-oriented surface $S^a \subset P$ which separates $\mathbf{v}^a$ from the other vertices. The boundary of S^a lies on those faces of P which meet at $\mathbf{v}^a$. Construct a new map $\boldsymbol{\nu}^a : S^a \to S^2$ which coincides with $\mathbf{n}$ on ∂S^a and whose image does not contain $\mathbf{s}$. The wrapping number $w^a(\mathbf{n})$ is the degree of the map $S^2 \to S^2$ obtained by gluing the maps $\mathbf{n}|_{S^a}$ and $\boldsymbol{\nu}^a$ along the boundary of S^a. The fact that $\mathbf{n}$ is continuous on P implies that $\sum_{a=1}^{v} w^a(\mathbf{n}) = 0$.

In what follows, we denote the invariants collectively by $\mathrm{inv} = \{\mathbf{e}^b, k^{ac}, w^a\}$.

The paper is organized as follows. A lower bound for the energies of C^∞ tangent unit-vector fields in terms of the invariants is given in Theorem 2.1. The derivation adapts methods of [5] to the tangent boundary-value problem treated here. (Boundary value problems were not considered in [5]. Also, the lower bound involves not only the degree of the map on two-dimensional surfaces surrounding the vertices, but also kink numbers and edge orientations). In Theorem 3.1 we establish that C^∞ tangent unit-vector fields are dense in the space of continuous Sobolev tangent unit-vector fields with respect to the Sobolev norm. The proof requires certain smoothings of the vector fields which lie outside the scope of the standard Meyers-Serrin theorem [6]. Thus, the lower bound of Theorem 2.1 extends to $C^0_{\mathfrak{T}}(P) \cap W^{1,2}(P)$. Section 4 contains a discussion of the results.

2 Lower bounds for energies of harmonic maps

Definition 2.1. The minimal energy $M(h)$ of maps in homotopy class h is defined by

$$M(h) = \inf_{\substack{\mathbf{n}\in C^0_{\mathfrak{T}}(P)\bigcap W^{1,2}(P),\\ \mathrm{inv}(\mathbf{n})=h}} E[\mathbf{n}], \tag{3}$$

where $W^{1,2}(P)$ is the Sobolev space

$$W^{1,2}(P) = \big\{\mathbf{n} \mid \nabla\mathbf{n} \in L^2(P)\big\} \tag{4}$$

and

$$E[\mathbf{n}] = \int\int\int_P (\nabla\mathbf{n})^2\, dV = \int\int\int_P \partial_a n_b \partial_a n_b\, dV = \|\mathbf{n}\|^2_{W^{1,2}(P)}. \tag{5}$$

If the infimum is actually achieved by some tangent unit-vector field $\mathbf{n} \in C^2_{\mathfrak{T}}(P)$, then $\mathbf{n}$ satisfies the Euler-Lagrange equation

$$\Delta\mathbf{n} - <\mathbf{n}, \Delta\mathbf{n}>\, \mathbf{n} = 0, \tag{6}$$

with boundary conditions $\mathbf{n}\cdot\mathbf{F}^c|_{F^c} = 0$, $((\mathbf{F}^c\cdot\nabla)\mathbf{n})\cdot(\mathbf{n}\times\mathbf{F}^c)|_{F^c} = 0$ on the faces of P. It will be shown in Section 3 that $C^\infty_{\mathfrak{T}}(P)\bigcap W^{1,2}(P)$ is dense in $C^0_{\mathfrak{T}}(P)\bigcap W^{1,2}(P)$ with respect to the Sobolev norm. Thus, to compute $M(h)$, it suffices to consider smooth maps only.

It is straightforward to show (a demonstration is given in [5]) that the energy density $\rho = (\nabla\mathbf{n})^2$ satisfies the inequality

$$\rho \geq 2\,|\mathbf{n}^*\omega|\,, \tag{7}$$

where ω is the area-form on S^2, normalized to have area 4π, and $|\mathbf{n}^*\omega|$ is the Euclidean norm of its pull-back. Indeed, since $(\mathbf{n}\cdot\nabla)\mathbf{n} = 0$, we may write

$$\nabla\mathbf{n} = \alpha_1\boldsymbol{\tau}_1 + \alpha_2\boldsymbol{\tau}_2, \tag{8}$$

where $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ constitute a (locally defined) orthonormal basis for $T_{\mathbf{n}}S^2$, and α_1, α_2 are (locally defined) one-forms on P. It follows that $\rho = |\alpha_1|^2 + |\alpha_2|^2$, while $|\mathbf{n}^*\omega| = |\alpha_1\wedge\alpha_2|^2 = |\alpha_1|^2|\alpha_2|^2 - (\alpha_1\cdot\alpha_2)^2$, where $\alpha_1\cdot\alpha_2$ denotes the Euclidean inner-product on forms. Therefore, $\rho^2 - 4\,|\mathbf{n}^*\omega|^2 = (|\alpha_1|^2 - |\alpha_2|^2)^2 + 4(\alpha_1\cdot\alpha_2)^2 \geq 0$.

Remark 2.1. Suppose $\alpha_1\wedge\alpha_2 \neq 0$. From (8), if $\iota_X(\alpha_1\wedge\alpha_2) = 0$ for some vector field X, then $\iota_X\nabla\mathbf{n} = 0$; ie, $\mathbf{n}$ is constant on the characteristics of $\alpha_1\wedge\alpha_2$.

Remark 2.2. If we have equality in (7), then $|\alpha_1|^2 = |\alpha_2|^2$ and $\alpha_1\cdot\alpha_2 = 0$, so that $\nabla\mathbf{n}$, regarded as a map from the orthgonal complement of the characteristic distribution of $\alpha_1\wedge\alpha_2$ to $T_{\mathbf{n}}S^2$, is conformal.

Definition 2.2. The *trapped area* $\Omega^a(\mathbf{n})$ at a vertex $\mathbf{v}^a$ is the area (as a proportion of the area of S^2) of the image under $\mathbf{n}$ of an outward-oriented surface, $S^a \subset P$, which separates $\mathbf{v}^a$ from the other vertices, ie

$$\Omega^a(\mathbf{n}) = \frac{1}{4\pi} \int\int_{S^a} \mathbf{n}^*\omega. \tag{9}$$

The trapped areas are homotopy invariants, and may be expressed in terms of edge orientations, kink numbers and wrapping numbers as follows (see [2] for details). Given a vertex $\mathbf{v}^a$, let K^a be the geodesic polygon on S^2 with vertices $\mathbf{e}^{b_1}, \ldots \mathbf{e}^{b_m}$ given by the edge orientations of the edges $E^{b_1}, \ldots, E^{b_m}$ which are incident at $\mathbf{v}^a$ (the edges are ordered consecutively with respect to the outward normal on S^a). Then

$$\Omega^a = w^a - \tfrac{1}{2}{\sum_c}' \operatorname{sgn}(\mathbf{F}^c \cdot \mathbf{s}) k^{ac} + \sum_{j=2}^{m-1} \left(\frac{1}{4\pi} A(\mathbf{e}^{b_1}, \mathbf{e}^{b_j}, \mathbf{e}^{b_{j+1}}) - \sigma(\mathbf{e}^{b_1}, \mathbf{e}^{b_j}, \mathbf{e}^{b_{j+1}}) \right), \tag{10}$$

where the sum ${\sum}'_c$ is taken over the faces F^c incident at $\mathbf{v}^a$, $A(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in (-2\pi, 2\pi)$ is the oriented area of the spherical triangle with vertices $\{\mathbf{a},\ \mathbf{b},\ \mathbf{c}\}$ and $\sigma(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is equal to $\operatorname{sgn}((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{s})$ if $\mathbf{s}$ is contained in the spherical triangle with vertices $\{\mathbf{a},\ \mathbf{b},\ \mathbf{c}\}$ and is zero otherwise. The trapped areas are typically not integer-valued. However, they satisfy the sum rule (related to the sum rule for the wrapping numbers)

$$\sum_a \Omega^a = 0 \tag{11}$$

(this follows from the fact that the map $\mathbf{n} : \partial P \to S^2$ is contractible).

Theorem 2.1. *The minimal energy $M(h)$ is bounded below by*

$$M(h) \geq \max_{\left\{\xi^a : |\xi^a - \xi^{a'}| \leq \left|\mathbf{v}^a - \mathbf{v}^{a'}\right|\right\}} \left(8\pi \sum_a \Omega^a \xi^a \right) > 0. \tag{12}$$

Remark 2.3. The bound is given in terms of a finite-dimensional linear optimization problem with linear constraints, whose solution can be found algorithmically using standard methods.

Proof. Let $\mathbf{n} \in C^\infty_{\mathfrak{T}}(P) \cap W^{1,2}(P)$ with $\operatorname{inv}(\mathbf{n}) = h$. Let $\xi \in \operatorname{Lip}_1(P)$, the space of Lipschitz functions on P with Lipschitz constant less then one. Then ξ is almost-everywhere differentiable with $|d\xi| \leq 1$. It follows from (7) that

$$E[\mathbf{n}] = \int\int\int_P \rho\, dV \geq 2 \int\int\int_P |\mathbf{n}^*\omega|\, dV \geq 2 \int\int\int_P d\xi \wedge \mathbf{n}^*\omega. \tag{13}$$

We remove infinitesimal neighbourhoods of the vertices from the domain of integration and integrate by parts in the last expression. As $d\,\mathbf{n}^*\omega = \mathbf{n}^*d\,\omega = 0$, the volume integral vanishes. Because $\mathbf{n}$ is tangent, $\mathbf{n}^*\omega$ vanishes on the faces F^c, so that the only contribution to the surface integral is from the boundaries of the excised infinitesimal neighbourhoods. On these boundaries, ξ can be replaced by its values at the vertices. Recalling (9), we obtain

$$\int\int\int_P d\xi \wedge \mathbf{n}^*\omega = 8\pi \sum_a \Omega^a \xi(\mathbf{v}^a). \tag{14}$$

Since (13) and (14) hold for all $\mathbf{n} \in C^\infty_{\mathfrak{T}}(P) \cap W^{1,2}(P)$, it follows that

$$M(h) \geq \sup_{\xi \in \mathrm{Lip}_1(P)} \left(8\pi \sum_a \Omega^a \xi(\mathbf{v}^a) \right). \tag{15}$$

$\xi \in \mathrm{Lip}_1(P)$ implies the constraints $|\xi(\mathbf{v}^a) - \xi(\mathbf{v}^{a'})| \leq |\mathbf{v}^a - \mathbf{v}^{a'}|$. Conversely, given a set of ξ^a's satisfying these constraints, we can construct a function $\xi \in \mathrm{Lip}_1(P)$ with $\xi(\mathbf{v}^a) = \xi^a$, eg by letting $\xi(\mathbf{r}) := \max_a (\xi^a - |\mathbf{r} - \mathbf{v}^a|)$. Therefore,

$$\sup_{\xi \in \mathrm{Lip}_1} \left(8\pi \sum_a \Omega^a \xi(\mathbf{v}^a) \right) = \max_{\left\{\xi^a : |\xi^a - \xi^{a'}| \leq |\mathbf{v}^a - \mathbf{v}^{a'}|\right\}} \left(8\pi \sum_a \Omega^a \xi^a \right), \tag{16}$$

which together with (15) gives the required lower bound. Note that (11) implies that the optimal ξ^a's are determined up to an additive constant, which we can fix, say, by setting $\xi^1 = 0$. The constraints ensure that the feasible set is nonempty and bounded; thus the maximal value in the right-hand side is finite. It is obvious that that maximal value is positive, so that the lower bound is nontrivial. □

Remark 2.4. The maximisation problem which appears in the lower bound (12) can be replaced by its equivalent dual minimisation problem, as in [5].

3 Approximating by smooth tangent maps

In Theorem 3.1 below we show that $C^\infty_{\mathfrak{T}}(P) \bigcap W^{1,2}(P)$ is dense in $C^0_{\mathfrak{T}}(P) \bigcap W^{1,2}(P)$. This is accomplished by constructing, for a given continuous tangent unit-vector field $\tilde{\mathbf{n}}$, a smooth tangent unit-vector field $\mathbf{n}$ arbitrarily close to $\tilde{\mathbf{n}}$ with respect to the Sobolev norm. Away from the vertices and edges of P, $\mathbf{n}$ is obtained from a smooth average of $\tilde{\mathbf{n}}$ which preserves tangent boundary conditions. Neighbourhoods of the vertices and edges require special treatment, which is dealt with in Lemmas 3.1 and 3.2 respectively.

Lemma 3.1. Vertex extension. *Let* $\mathbf{v}^a$ *be a vertex of* P*. Let* l^a *be a ray from* $\mathbf{v}^a$ *into the interior of* P*, and introduce local Euclidean coordinates centred at* $\mathbf{v}^a$ *with positive* z*-axis along* l^a*. Let*

$$\Lambda^a(H) := \{(x, y, z) \in P \,|\, 0 < z \leq H\} \tag{17}$$

denote the prism obtained by cutting P *by a plane perpendicular to* l^a *at a distance* H *from* $\mathbf{v}^a$*. Let*

$$\Pi^a = \overline{\Lambda^a(H) \setminus \Lambda^a(\tfrac{1}{2}H)} \tag{18}$$

denote the closed lower half of $\Lambda^a(H)$*. Given a* C^∞ *tangent unit-vector field* $\tilde{\mathbf{n}}$ *in* Π^a *and some* $\epsilon_3 > 0$ *such that*

$$E[\tilde{\mathbf{n}}] := \int\int\int_{\Pi^a} (\nabla\tilde{\mathbf{n}})^2 \, dV \leq \epsilon_3, \tag{19}$$

one can construct a C^∞ *tangent unit-vector field* $\mathbf{n}$ *on* $\Lambda^a(H)$ *coinciding with* $\tilde{\mathbf{n}}$ *on* $\Lambda^a(H) \setminus \Lambda^a(\frac{3}{4}H)$ *such that*

$$E[\mathbf{n}] := \int\int\int_{\Lambda^a(H)} (\nabla\mathbf{n})^2 \, dV \leq C^a\epsilon_3, \tag{20}$$

where $C^a > 0$ *is independent of* $\tilde{\mathbf{n}}$*,* ϵ_3 *and* H*.*

Proof. Let $D := \{(u, v) \,|\, (Hu, Hv, H) \in P\}$ denote the base of $\Lambda^a(H)$, parameterized by u and v. Introduce new coordinates (u, v, h) by

$$z = Hh, \quad x = Hhu, \quad y = Hhv, \quad \text{where } 0 < h \leq 1, \text{ and } (u, v) \in D. \tag{21}$$

Let $\tilde{\boldsymbol{\phi}}(u, v, h) = \tilde{\mathbf{n}}(Hhu, Hhv, Hh)$. From (19),

$$E[\tilde{\mathbf{n}}] = H\int_{\frac{1}{2}}^{1} dh \int\int_D dudv \left\{ \left(\tilde{\boldsymbol{\phi}}_u\right)^2 + \left(\tilde{\boldsymbol{\phi}}_v\right)^2 + \left(-u\tilde{\boldsymbol{\phi}}_u - v\tilde{\boldsymbol{\phi}}_v + h\tilde{\boldsymbol{\phi}}_h\right)^2 \right\} \leq \epsilon_3. \tag{22}$$

Then

$$\|\tilde{\boldsymbol{\phi}}_u\|^2_{L^2(\Pi^a)} \leq \epsilon_3/H, \quad \|\tilde{\boldsymbol{\phi}}_v\|^2_{L^2(\Pi^a)} \leq \epsilon_3/H, \tag{23}$$

where $\|\mathbf{a}\|_{L^2(\Pi^a)}$ means $\left(\int_{1/2}^1 dh \int\int_D dudv\, |\mathbf{a}|^2\right)^{\frac{1}{2}}$. Since u and v are bounded on D, it follows that

$$\tfrac{1}{2}\|\tilde{\boldsymbol{\phi}}_h\|_{L^2(\Pi^a)} \leq \|h\tilde{\boldsymbol{\phi}}_h\|_{L^2(\Pi^a)} \leq (\epsilon_3/H)^{\frac{1}{2}} + \|u\tilde{\boldsymbol{\phi}}_u\|_{L^2(\Pi^a)} + \|v\tilde{\boldsymbol{\phi}}_v\|_{L^2(\Pi^a)} \leq C_1^a(\epsilon_3/H)^{\frac{1}{2}} \tag{24}$$

for some $C_1^a > 0$ independent of ϵ_3, H and $\tilde{\mathbf{n}}$.

$\tilde{\boldsymbol{\phi}}$ may be extended to a unit-vector field $\boldsymbol{\phi}$ on $\Lambda^a(H)$ according to

$$\boldsymbol{\phi}(u, v, h) = \tilde{\boldsymbol{\phi}}(u, v, s(h)), \quad 0 < h \leq 1, \tag{25}$$

where $s(h)$ is a C^∞ function on $[0,1]$ with $\frac{1}{2} \le s(h) \le 1$, $s(0) = \frac{1}{2}$, $s(h) = h$ for $\frac{3}{4} \le h \le 1$, and with $s'(h)$ bounded away from zero. For example, we can take

$$\begin{aligned}
& s(h) = \tfrac{3}{4} + \textstyle\int_{\frac{3}{4}}^{h} \alpha(t)dt, \\
& \alpha(x) := a + (1-a)\frac{\int_0^x f(t)dt}{\int_0^{\frac{3}{4}} f(t)dt}, \\
& f(x) := \begin{cases} \exp\left[\frac{1}{x-\frac{3}{4}} - \frac{1}{x-\frac{1}{2}}\right], x \in (\frac{1}{2}, \frac{3}{4}) \\ 0, \quad x \notin (\frac{1}{2}, \frac{3}{4}) \end{cases}, \\
& a := \frac{\gamma - \frac{1}{4}}{\gamma - \frac{3}{4}}, \gamma := \frac{\int_{\frac{1}{2}}^{\frac{3}{4}} f(t)(\frac{3}{4}-t)dt}{\int_{\frac{1}{2}}^{\frac{3}{4}} f(t)dt} < \tfrac{1}{4}.
\end{aligned} \tag{26}$$

We define the corresponding extension of $\tilde{\mathbf{n}}$ by

$$\mathbf{n}(x,y,z) = \boldsymbol{\phi}\left(\frac{x}{z}, \frac{y}{z}, \frac{z}{H}\right), \quad 0 < z \le H. \tag{27}$$

Clearly, $\mathbf{n}$ is a smooth tangent unit-vector field on $\Lambda^a(H)$. We estimate its energy (20) as follows. Let $h(s)$ be the inverse of $s(h)$. Since $h^2(s)s'(h(s))$ and $1/s'(h(s))$ are bounded on $[0,1]$, say by C_2,

$$\begin{aligned}
E[\mathbf{n}] &= H\int_0^1 dh \int_D dudv \left\{(\boldsymbol{\phi}_u)^2 + (\boldsymbol{\phi}_v)^2 + (-u\boldsymbol{\phi}_u - v\boldsymbol{\phi}_v + h\boldsymbol{\phi}_h)^2\right\} = \\
&= H\int_{\frac{1}{2}}^1 \frac{ds}{s'(h(s))}\int_D dudv \left\{\left(\tilde{\boldsymbol{\phi}}_u\right)^2 + \left(\tilde{\boldsymbol{\phi}}_v\right)^2 + \left(-u\tilde{\boldsymbol{\phi}}_u - v\tilde{\boldsymbol{\phi}}_v + h(s)s'(h(s))\tilde{\boldsymbol{\phi}}_h\right)^2\right\} \le \\
&\le C_2 H\int_{\frac{1}{2}}^1 dh\int_D dudv\left\{\left(\tilde{\boldsymbol{\phi}}_u\right)^2 + \left(\tilde{\boldsymbol{\phi}}_v\right)^2 + 3\left(u\tilde{\boldsymbol{\phi}}_u\right)^2 + 3\left(v\tilde{\boldsymbol{\phi}}_v\right)^2 + 3\left(\tilde{\boldsymbol{\phi}}_h\right)^2\right\}.
\end{aligned} \tag{28}$$

In the last step of (28), we have used the elementary inequality $(a+b+c)^3 \le 3a^3+3b^3+3c^3$ (similar inqualities are used in what follows). Using (23), (24) and the fact that u and v are bounded on D, we obtain

$$E[\mathbf{n}] \le C_3^a H\left(\|\tilde{\boldsymbol{\phi}}_u\|^2_{L^2(\Pi^a)} + \|\tilde{\boldsymbol{\phi}}_v\|^2_{L^2(\Pi^a)} + \|\tilde{\boldsymbol{\phi}}_h\|^2_{L^2(\Pi^a)}\right) \le C^a\epsilon_3, \tag{29}$$

for some $C_3^a > 0$ and $C^a > 0$ independent of ϵ_3, H and $\tilde{\mathbf{n}}$. $\square$

Lemma 3.2. Edge extension. *Let E^b be an edge of P between faces F^+ and F^-. Let (x,y,z) denote local Euclidean coordinates centred about the midpoint of E^b, with z-axis parallel to E^b and positive y-axis directed into P and lying in the midplane between F^+ and F^-, so that $F^\pm$ are given locally by $y \ge 0, x = \pm\tau y$, where $\tau = \tan\alpha$ and 2α is the angle between F^+ and F^-. Let*

$$\Lambda^b(W,L) := \left\{(x,y,z)\,|\, 0 \le y \le W, -\tau y \le x \le \tau y, -\tfrac{1}{2}L \le z \le \tfrac{1}{2}L\right\} \tag{30}$$

denote the right prism whose axis has length L, is parallel to E^b, and is symmetric about the midpoint of E^b, and whose cross-section is an isosceles triangle with sides in F^+ and F^- of length $W \sec\alpha$. Let

$$\Pi^b = \overline{\Lambda^b(W,L) \setminus \Lambda^b(\tfrac{1}{2}W,L)} \tag{31}$$

denote the closed interior half of this prism. Let $\mathbf{e}^b$ be a unit vector parallel to E^b. Then, given a C^∞ unit-vector field $\tilde{\mathbf{n}}$ on Π^b which satisfies tangent boundary conditions on $F^\pm$, and constants $\epsilon_1 > 0$ and $0 < \epsilon_2 < \frac{1}{2}$ such that

$$E[\tilde{\mathbf{n}}] := \int_{\Pi^b} (\nabla\tilde{\mathbf{n}})^2 \leq \epsilon_1, \tag{32}$$

$$\max_{\mathbf{r}\in\Pi^b} |\tilde{\mathbf{n}}(\mathbf{r}) - \mathbf{e}^b| \leq \epsilon_2, \tag{33}$$

one can construct a C^∞ unit-vector field $\mathbf{n}$ in $\Lambda^b(W,L)$ satisfying tangent boundary conditions on $F^\pm$ and coinciding with $\tilde{\mathbf{n}}$ on $\Lambda^b(W,L) \setminus \Lambda^b(\frac{3}{4}W,L)$ such that

$$E[\mathbf{n}] := \int_{\Lambda^b(W,L)} (\nabla\mathbf{n})^2 \leq C^b(\epsilon_1 + L{\epsilon_2}^2), \tag{34}$$

where C^b is independent of $\tilde{\mathbf{n}}$, ϵ_1, ϵ_2, L and W.

Proof. It is convenient to introduce new coordinates (ξ, η, μ) as follows:

$$y = W\eta, \quad x = \tau W\eta\xi, \quad z = L\mu. \tag{35}$$

Let

$$\tilde{\boldsymbol{\phi}}(\xi,\eta,\mu) = \tilde{\mathbf{n}}(\tau W\eta\xi, W\eta, L\mu). \tag{36}$$

Then

$$E[\tilde{\mathbf{n}}] = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\mu \int_{-1}^{1} d\xi \int_{\frac{1}{2}}^{1} \frac{d\eta}{\eta} \left\{ \frac{L}{\tau}(\tilde{\boldsymbol{\phi}}_\xi)^2 + L\tau(\eta\tilde{\boldsymbol{\phi}}_\eta - \xi\tilde{\boldsymbol{\phi}}_\xi)^2 + \frac{W^2\tau}{L}\eta^2(\tilde{\boldsymbol{\phi}}_\mu)^2 \right\} \leq \epsilon_1. \tag{37}$$

Let $\|\mathbf{a}\|_{L^2(\Pi^b)}$ denote the L^2-norm $\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} d\mu \int_{-1}^{1} d\xi \int_{\frac{1}{2}}^{1} d\eta\, |\mathbf{a}|^2\right)^{\frac{1}{2}}$ of a vector field $\mathbf{a}$ on Π^b. From (37) one can deduce that

$$\|\tilde{\boldsymbol{\phi}}_\xi\|^2_{L^2(\Pi^b)} \leq C_1^b\frac{\epsilon_1}{L}, \quad \|\tilde{\boldsymbol{\phi}}_\eta\|^2_{L^2(\Pi^b)} \leq C_1^b\frac{\epsilon_1}{L}, \quad \|\tilde{\boldsymbol{\phi}}_\mu\|^2_{L^2(\Pi^b)} \leq C_1^b\frac{L\epsilon_1}{W^2} \tag{38}$$

for some $C_1^b > 0$ independent of $\tilde{\mathbf{n}}$, ϵ_1, ϵ_2, L and W (in fact, it suffices to take $C_1^b = 4(\tau^{\frac{1}{2}} + \tau^{-\frac{1}{2}})^2$).

We extend $\tilde{\boldsymbol{\phi}}$ to a tangent unit-vector field $\boldsymbol{\phi}$ on $\Lambda^b(W,L)$ as follows. In general, let $a_\parallel = \mathbf{a}\cdot\mathbf{e}^b$ and $\mathbf{a}_\perp = \mathbf{a} - a_\parallel \mathbf{e}^b$ denote the components of $\mathbf{a}$ parallel and perpendicular to $\mathbf{e}^b$. Let $\psi(\eta)$ be a C^∞ map of the unit interval $[0,1]$ into itself such that $\psi(\eta) = 1$ for $0 \le \eta \le \frac{1}{4}$ and $\psi(\eta) = 0$ for $\frac{1}{2} \le \eta \le 1$. Let $\Gamma(\xi,\eta,\mu) = (\xi, s(\eta), \mu)$, ie Γ denotes the change of coordinates $\eta \mapsto s(\eta)$ which leaves ξ and μ unchanged, where $s(\eta)$ is a function like the one described in (26). Then $\boldsymbol{\phi}$ is given by

$$\boldsymbol{\phi}_\perp = (1-\psi)\left(\tilde{\boldsymbol{\phi}}_\perp \circ \Gamma\right), \quad \phi_\parallel = \left(1 - |\boldsymbol{\phi}_\perp|^2\right)^{\frac{1}{2}}. \tag{39}$$

From (33) it follows that $|\tilde{\boldsymbol{\phi}}_\perp| \le \epsilon_2 \le \frac{1}{2}$ in Π^b, so that $\phi_\parallel$ is smooth.

The corresponding extenson of $\tilde{\mathbf{n}}$ is given by

$$\mathbf{n}(x,y,z) = \boldsymbol{\phi}\left(\frac{x}{\tau y}, \frac{y}{W}, \frac{z}{L}\right). \tag{40}$$

In analogy with (37), the energy of $\mathbf{n}$ is given by

$$E[\mathbf{n}] = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\mu \int_{-1}^{1} d\xi \int_0^1 \frac{d\eta}{\eta} \left\{ \frac{L}{\tau}\left(\boldsymbol{\phi}_\xi\right)^2 + L\tau\left(\eta\boldsymbol{\phi}_\eta - \xi\boldsymbol{\phi}_\xi\right)^2 + \frac{W^2\tau}{L}\eta^2\left(\boldsymbol{\phi}_\mu\right)^2 \right\}. \tag{41}$$

Noting that $\boldsymbol{\phi} = \mathbf{e}^b$ for $\eta \le \frac{1}{4}$ (so that integrand vanishes in this range), one can obtain the estimate

$$E[\mathbf{n}] \le C_2^b \int_{-\frac{1}{2}}^{\frac{1}{2}} d\mu \int_{-1}^{1} d\xi \int_0^1 d\eta \left\{ L\left(\boldsymbol{\phi}_\xi\right)^2 + L\left(\boldsymbol{\phi}_\eta\right)^2 + \frac{W^2}{L}\left(\boldsymbol{\phi}_\mu\right)^2 \right\} \tag{42}$$

for some $C_2^b > 0$ independent of $\tilde{\mathbf{n}}$, ϵ_1, ϵ_2, L and W (in fact, it suffices to take $C_2^b = 8\tau + 4\tau^{-1}$).

The derivatives of $\boldsymbol{\phi}$ can be expressed in terms of derivatives of $\tilde{\boldsymbol{\phi}}$ as follows. We have that $\boldsymbol{\phi}_{\perp\xi} = (1-\psi)(\tilde{\boldsymbol{\phi}}_{\perp\xi} \circ \Gamma)$, while $\phi_{\parallel\xi} = (1-\psi)^2((\tilde{\phi}_\parallel\, \tilde{\phi}_{\parallel\xi}) \circ \Gamma)/\phi_\parallel$, where we have used $|\tilde{\boldsymbol{\phi}}_\perp|^2 + \tilde{\phi}_\parallel^2 = 1$. Since $|1-\psi|^2 \le 1$ and $|(\tilde{\phi}_\parallel \circ \Gamma)/\phi_\parallel| \le 1$, it follows that

$$(\boldsymbol{\phi}_\xi)^2 = (\boldsymbol{\phi}_{\perp\xi})^2 + (\phi_{\parallel\xi})^2 \le (\tilde{\boldsymbol{\phi}}_\xi \circ \Gamma)^2. \tag{43}$$

A similar argument shows that

$$(\boldsymbol{\phi}_\mu)^2 \le (\tilde{\boldsymbol{\phi}}_\mu \circ \Gamma)^2. \tag{44}$$

Next, we have that

$$\boldsymbol{\phi}_{\perp\eta} = -\psi'(\tilde{\boldsymbol{\phi}}_\perp \circ \Gamma) + (1-\psi)\, s'\,(\tilde{\boldsymbol{\phi}}_{\perp\eta} \circ \Gamma), \tag{45}$$

which implies that

$$(\boldsymbol{\phi}_{\perp\eta})^2 \le 2\psi'^2|\tilde{\boldsymbol{\phi}}_\perp \circ \Gamma|^2 + 2s'^2(\tilde{\boldsymbol{\phi}}_{\perp\eta} \circ \Gamma)^2. \tag{46}$$

Similarly,

$$\phi_{\|\eta} = (1-\psi)\psi' \left(|\tilde{\boldsymbol{\phi}}_\perp|^2 \circ \Gamma\right)/\phi_\| + (1-\psi)^2\, s'\,((\tilde{\phi}_\|\,\tilde{\phi}_{\|\eta}) \circ \Gamma)/\phi_\|. \tag{47}$$

From (33), $|\tilde{\boldsymbol{\phi}}_\perp|^2 \le {\epsilon_2}^2 < \frac{1}{4}$ and $|\phi_\|| > \frac{1}{2}$, so that

$$(\phi_{\|\eta})^2 \le 2{\psi'}^2{\epsilon_2}^2 + 2{s'}^2(\tilde{\phi}_{\|\eta} \circ \Gamma)^2. \tag{48}$$

Together, (46) and (48) give

$$(\boldsymbol{\phi}_\eta)^2 \le 4C_3{\epsilon_2}^2 + 2{s'}^2(\tilde{\boldsymbol{\phi}}_\eta \circ \Gamma)^2, \tag{49}$$

where C_3 bounds ${\psi'}^2$.

We substitute (43), (44) and (49) into the estimate (42). Replacing $\int_0^1 d\eta$ by $\int_{\frac{1}{2}}^1 \eta'(s)ds$, we get

$$E[\mathbf{n}] \le C_2^b C_4 \left(L\|\tilde{\boldsymbol{\phi}}_\xi\|^2_{L^2(\Pi^b)} + L\left(4C_3{\epsilon_2}^2 + 2\|\tilde{\boldsymbol{\phi}}_\eta\|^2_{L^2(\Pi^b)}\right) + \frac{W^2}{L}\|\tilde{\boldsymbol{\phi}}_\mu\|^2_{L^2(\Pi^b)}\right), \tag{50}$$

where C_4 bounds both s' and $\eta' = 1/s'$. From (38),

$$E[\mathbf{n}] \le 4C_1^b C_2^b C_4 \epsilon_1 + 4LC_2^b C_3 C_4 {\epsilon_2}^2, \tag{51}$$

which implies the required result. $\square$

Theorem 3.1. *Suppose the tangent unit-vector field $\tilde{\mathbf{n}}$ is continuous on P with $\|\tilde{\mathbf{n}}\|^2_{1,2}(P) = \int_P (\nabla\tilde{\mathbf{n}})^2 < \infty$. Then for all $\epsilon > 0$, there exists a $C^\infty(P)$ tangent unit-vector field $\mathbf{n}$ homotopic to $\tilde{\mathbf{n}}$ with $\|\tilde{\mathbf{n}} - \mathbf{n}\|_{1,2}(P) \le \epsilon$.*

Remark 3.1. In general, $\mathbf{n}(\mathbf{r}) - \tilde{\mathbf{n}}(\mathbf{r})$ is not uniformly small in $\mathbf{r}$; in small neighbourhoods of the edges and vertices, this difference may be of order one. However, the contribution of these neighbourhoods to the Sobolev norm, $\|\tilde{\mathbf{n}} - \mathbf{n}\|_{1,2}(P)$, is small.

Proof. Step 0: truncation, reflection. Let $\Lambda^a(H)$ denote the vertex prisms defined in Lemma 3.1. We choose H sufficiently small so that these do not intersect, and so that, for given $\epsilon_1 > 0$,

$$\|\tilde{\mathbf{n}}\|^2_{1,2}\left(\cup_a \Lambda^a(H)\right) \le \epsilon_1. \tag{52}$$

Let $\Lambda^b(W, \tilde{L}^b)$ denote the edge prisms defined in Lemma 3.2. We take these to have the same width, W, but allow them to have different lengths $\tilde{L}^b$. We choose W sufficiently small and the $\tilde{L}^b$'s so that the following conditions are satisfied: First, the $\Lambda^b(W, \tilde{L}^b)$'s do not interesect. Second, the top and bottom faces of $\Lambda^b(W, \tilde{L}^b)$ (given by $z = \pm\frac{1}{2}\tilde{L}^b$ in the local Euclidean coordinates of (30)) are contained in the respective vertex prisms $\Lambda^a(\frac{3}{12}H)$ and

$\Lambda^{a'}(\frac{3}{12}H)$ at the endpoints of E^b, but do not intersect the smaller vertex prisms $\Lambda^a(\frac{2}{12}H)$ and $\Lambda^{a'}(\frac{2}{12}H)$. Third, for given $\epsilon_1 > 0$,

$$\|\tilde{\mathbf{n}}\|^2_{1,2}\left(\cup_b \Lambda^b(W, \tilde{L}^b)\right) \leq \tfrac{1}{2}\epsilon_1. \tag{53}$$

Fourth, for given $0 < \epsilon_2 < \frac{1}{2}$,

$$\max_{\mathbf{r}\in\Lambda^b(W,\tilde{L}^b)} \left|\tilde{\mathbf{n}}(\mathbf{r}) - \mathbf{e}^b\right| \leq \tfrac{1}{2}\epsilon_2 < \tfrac{1}{4}, \tag{54}$$

where $\mathbf{e}^b$ is the edge orientation of $\tilde{\mathbf{n}}$ on E^b.

Let

$$P' = \overline{P \setminus \left(\cup_a \Lambda^a(\tfrac{4}{12}H) \cup_b \Lambda^b(\tfrac{1}{2}W, \tilde{L}^b)\right)} \tag{55}$$

be the closed polyhedron obtained by removing vertex and edge prisms of indicated size from P. Choose $\Delta_1 > 0$ sufficiently small so that, for all $\mathbf{r} \in P'$, $B(\mathbf{r}, \Delta_1)$ – the Δ_1-ball centered at $\mathbf{r}$ – intersects at most one face of P. Let

$$P' + \Delta_1 = \left\{\mathbf{r} \in \mathbb{R}^3 \middle| \, |\mathbf{r} - \mathbf{r}'| \leq \Delta_1 \text{ for some } \mathbf{r}' \in P'\right\} \tag{56}$$

denote the closed Δ_1-neighbourhood of P'. We define a unit-vector field $\tilde{\mathbf{n}}_+$ on $P' + \Delta_1$ as follows. If $\mathbf{r} \in P$, $\tilde{\mathbf{n}}_+(\mathbf{r})$ is just taken to be $\tilde{\mathbf{n}}(\mathbf{r})$. If $\mathbf{r} \in (P' + \Delta_1) \setminus P$, then we must have that $\mathbf{r} = \mathbf{p} + \alpha\mathbf{F}^c$ for some (uniquely determined) $\mathbf{p}$ on a face F^c of P and for some $0 < \alpha \leq \Delta_1$. In this case, we take $\tilde{\mathbf{n}}_+(\mathbf{p} + \alpha\mathbf{F}^c) = \mathcal{R} \cdot \tilde{\mathbf{n}}_+(\mathbf{p} - \alpha\mathbf{F}^c)$, where $\mathcal{R}$ denotes reflection about the plane normal to $\mathbf{F}^c$. It is clear that the Sobolev norm $\|\tilde{\mathbf{n}}_+\|_{1,2}(P' + \Delta_1)$ of the extended vector field is finite. Tangent boundary conditions imply that $\tilde{\mathbf{n}}_+$ is continuous at points $\mathbf{p} \in P'$ that lie on a face of P. Moreover, the average of $\tilde{\mathbf{n}}_+$ over $B(\mathbf{p}, \delta)$, where $\delta < \Delta_1$, is tangent to the face. This remains true for weighted averages over $B(\mathbf{p}, \delta)$ provided the weights at $\mathbf{p} + \alpha\mathbf{F}^c$ and $\mathbf{p} - \alpha\mathbf{F}^c$ are equal.

Step 1: Bulk. Since $\tilde{\mathbf{n}}$ is continuous on P, $\tilde{\mathbf{n}}_+$ is continuous on $P' + \Delta_1$, and therefore uniformly continuous (since $P' + \Delta_1$ is compact). Choose $\Delta_2 > 0$ sufficiently small so that

$$\mathbf{r}, \mathbf{r}' \in P' + \Delta_1, \quad |\mathbf{r} - \mathbf{r}'| \leq \Delta_2 \implies |\tilde{\mathbf{n}}_+(\mathbf{r}) - \tilde{\mathbf{n}}_+(\mathbf{r}')| \leq \tfrac{1}{4}\epsilon_2. \tag{57}$$

Let $K(r) \geq 0$ be a smooth function with support contained in $(0, 1)$, normalized so that

$$4\pi \int_0^\infty K(r) r^2 dr = 1. \tag{58}$$

Choose $\delta > 0$ so that $\delta < \min(\Delta_1, \Delta_2)$. We construct a smooth unit-vector field $\mathbf{u}$ on P' by averaging $\tilde{\mathbf{n}}_+$ over balls of radius δ, as follows:

$$\mathbf{u}(\mathbf{r}) = \int_{\mathbb{R}^3} \frac{1}{\delta^3} K\left(\frac{|\mathbf{r} - \mathbf{r}'|}{\delta}\right) \tilde{\mathbf{n}}_+(\mathbf{r}')\, dV'. \tag{59}$$

Uniform continuity ensures that $|\mathbf{u}(\mathbf{r}) - \tilde{\mathbf{n}}(\mathbf{r})| \le \frac{1}{4}\epsilon_2 < \frac{1}{8}$ for $\mathbf{r} \in P'$, and in particular that $\mathbf{u} \neq 0$. Therefore, the unit-vector field

$$\mathbf{v} := \mathbf{u}/|\mathbf{u}| \tag{60}$$

is smooth, with

$$\max_{\mathbf{r}\in P'} |\mathbf{v}(\mathbf{r}) - \tilde{\mathbf{n}}(\mathbf{r})| \le \tfrac{1}{2}\epsilon_2. \tag{61}$$

$\mathbf{v}$ satisfies tangent boundary conditions at points $\mathbf{p}$ in P' on faces F^c of P (since $K(\alpha\mathbf{F}^c/\delta) = K(-\alpha\mathbf{F}^c/\delta)$).

It is straightforward to show (the arguments are similar to those of the Meyers-Serrin theorem [6], see also Appendix A, [5]) that $\|\mathbf{v}-\tilde{\mathbf{n}}\|_{1,2}(P')$ can be made arbitrarily small with δ. Indeed, from (59), for $\mathbf{r} \in P'$,

$$\begin{aligned}\nabla\mathbf{u}(\mathbf{r}) - \nabla\tilde{\mathbf{n}}(\mathbf{r}) &= \int_{\mathbb{R}^3} \frac{1}{\delta^3} K\left(\frac{|\mathbf{r}-\mathbf{r}'|}{\delta}\right) \left(\nabla\tilde{\mathbf{n}}_+(\mathbf{r}') - \nabla\tilde{\mathbf{n}}_+(\mathbf{r})\right) \, d^3r' = \\ &= \int_{\mathbb{R}^3} \left[\frac{1}{\delta^{3/2}} K^{\frac{1}{2}}\left(\frac{|\mathbf{r}-\mathbf{r}'|}{\delta}\right)\right] \left[\frac{1}{\delta^{3/2}} K^{\frac{1}{2}}\left(\frac{|\mathbf{r}-\mathbf{r}'|}{\delta}\right) \left(\nabla\tilde{\mathbf{n}}_+(\mathbf{r}') - \nabla\tilde{\mathbf{n}}_+(\mathbf{r})\right)\right] \, d^3r'.\end{aligned} \tag{62}$$

The Cauchy-Schwartz inequality gives that

$$|\nabla\mathbf{u}(\mathbf{r}) - \nabla\tilde{\mathbf{n}}(\mathbf{r})|^2 \le \int_{\mathbb{R}^3} \frac{1}{\delta^3} K\left(\frac{|\mathbf{r}-\mathbf{r}'|}{\delta}\right) \left|\nabla\tilde{\mathbf{n}}_+(\mathbf{r}) - \nabla\tilde{\mathbf{n}}_+(\mathbf{r}')\right|^2 \, d^3r'. \tag{63}$$

Since square-integrable vector fields can be approximated arbitrarily closely (with respect to the L^2-norm) by continuous vector fields, we may write $\nabla\tilde{\mathbf{n}}_+ = \mathbf{c}+\mathbf{h}$, where $\mathbf{c}$ is continuous, and therefore uniformly continuous, on $P' + \Delta_1$, and $\|\mathbf{h}\|_{L^2}(P' + \Delta_1)$ is arbitrarily small. Extend $\mathbf{h}$ by zero outside $P' + \Delta_1$. Then

$$\begin{aligned}\|\nabla\mathbf{u} - \nabla\tilde{\mathbf{n}}_+\|^2_{L^2}(P') &\le \int_{P'} d^3r \int_{\mathbb{R}^3} d^3r' \frac{1}{\delta^3} K\left(\frac{|\mathbf{r}'-\mathbf{r}|}{\delta}\right) \left|\nabla\tilde{\mathbf{n}}_+(\mathbf{r}) - \nabla\tilde{\mathbf{n}}_+(\mathbf{r}')\right|^2 \le \\ &\le \int_{P'} d^3r \int_{\mathbb{R}^3} d^3r' \frac{1}{\delta^3} K\left(\frac{|\mathbf{r}-\mathbf{r}'|}{\delta}\right) 3\left(|\mathbf{h}(\mathbf{r})|^2 + |\mathbf{h}(\mathbf{r}')|^2 + |\mathbf{c}(\mathbf{r}) - \mathbf{c}(\mathbf{r}')|^2\right) \le \\ &\le 3\|\mathbf{h}\|^2_{L^2}(P'+\Delta_1) + 3\|\mathbf{h}\|^2_{L^2}(P') + 3\int_{P'} d^3r \int_{\mathbb{R}^3} d^3r' \frac{1}{\delta^3} K\left(\frac{|\mathbf{r}-\mathbf{r}'|}{\delta}\right) |\mathbf{c}(\mathbf{r}) - \mathbf{c}(\mathbf{r}')|^2 .\end{aligned}$$

The first two terms can be made arbitrarily small, and since $\mathbf{c}$ is uniformly continuous, the last term approaches zero with δ. That the same is true for $\mathbf{v} = \mathbf{u}/|\mathbf{u}|$ is easily established; note that $\nabla|\mathbf{u}|^2 = \nabla\left((\mathbf{u}-\tilde{\mathbf{n}}_+)\cdot(\mathbf{u}+\tilde{\mathbf{n}}_+)\right)$, and that $(\mathbf{u}+\tilde{\mathbf{n}}_+)$ is uniformly bounded on P' while $(\mathbf{u}-\tilde{\mathbf{n}}_+)$ approaches zero uniformly with δ. Thus, we can choose δ small enough so that

$$\|\mathbf{v}-\tilde{\mathbf{n}}\|^2_{1,2}(P') \le \tfrac{1}{2}\epsilon_1. \tag{64}$$

Step 2: Edges. Let $\Lambda^b(W, L^b)$ be edge prisms shorter than those introduced in Step 0, ie $L^b < \tilde{L}^b$, such that their top and bottom faces are contained in $\cup_a \Lambda^a(\frac{5}{12}H) \setminus \Lambda^a(\frac{4}{12}H)$. Let $\Pi^b = \overline{\Lambda^b(W, L^b) \setminus \Lambda^b(\frac{1}{2}W, L^b)}$ denote the half-prism, as in Lemma 3.2, and let

$$\Lambda_E = \cup_b \Lambda^b(W, L^b), \quad \Pi_E = \cup_b \Pi^b \tag{65}$$

denote the union of the edge prisms and half-prisms respectively. $\mathbf{v}$ defines a smooth tangent unit-vector field on Π_E. From (53) and (64), its energy is bounded by

$$\|\mathbf{v}\|^2_{1,2}(\Pi_E) \le 2\|\mathbf{v} - \tilde{\mathbf{n}}\|^2_{1,2}(\Pi_E) + 2\|\tilde{\mathbf{n}}\|^2_{1,2}(\Pi_E) \le \epsilon_1 + \epsilon_1 \le 2\epsilon_1. \tag{66}$$

From (54) and (61),

$$\max_{\mathbf{r}\in\Pi^b} \left|\mathbf{v}(\mathbf{r}) - \mathbf{e}^b\right| \le \max_{\mathbf{r}\in\Pi^b} \left(|\mathbf{v}(\mathbf{r}) - \tilde{\mathbf{n}}(\mathbf{r})| + \left|\tilde{\mathbf{n}}(\mathbf{r}) - \mathbf{e}^b\right|\right) \le \tfrac{1}{2}\epsilon_2 + \tfrac{1}{2}\epsilon_2 = \epsilon_2. \tag{67}$$

Thus, the conditions of Lemma 3.2 are satisfied for each $\Lambda^b(W, L^b)$, and we can construct a smooth tangent unit-vector field $\mathbf{w}$ on Λ_E which coincides with $\mathbf{v}$ on $\Lambda_E \setminus (\cup_b \Lambda^b(\frac{3}{4}W, L^b))$ such that

$$\|\mathbf{w}\|^2_{1,2}(\Lambda_E) \le \textstyle\sum_b C^b(2\epsilon_1 + L^b{\epsilon_2}^2), \tag{68}$$

where $C^b > 0$ is independent of ϵ_1, ϵ_2, $\tilde{\mathbf{n}}$, W and L^b. We may extend $\mathbf{w}$ (smoothly) to $P'' = \overline{P \setminus \left(\cup_a \Lambda^a(\frac{5}{12}H)\right)}$ by taking $\mathbf{w}$ to coincide with $\mathbf{v}$ on $P'' \setminus \Lambda_E$. From (53), (64) and (68),

$$\begin{aligned}
\|\mathbf{w} - \tilde{\mathbf{n}}\|^2_{1,2}(P'') &= \|\mathbf{w} - \tilde{\mathbf{n}}\|^2_{1,2}(P'' \setminus \Lambda_E) + \|\mathbf{w} - \tilde{\mathbf{n}}\|^2_{1,2}(\Lambda_E) \\
&\le \|\mathbf{v} - \tilde{\mathbf{n}}\|^2_{1,2}(P'' \setminus \Lambda_E) + 2\left(\|\mathbf{w}\|^2_{1,2} + \|\tilde{\mathbf{n}}\|^2_{1,2}\right)(\Lambda_E) \\
&\le \left(\tfrac{3}{2} + 4\textstyle\sum_b C^b\right)\epsilon_1 + 2\textstyle\sum_b C^b L^b {\epsilon_2}^2.
\end{aligned} \tag{69}$$

In the second inequality we have used the fact that $P'' \setminus \Lambda_E \subset P'$.

Step 3. Vertices. Let $\Pi^a = \overline{\Lambda^a(H) \setminus \Lambda^a(\frac{1}{2}H)}$ denote the half-prism, as in Lemma 3.1, and let

$$\Lambda_V = \cup_a \Lambda^a(H), \quad \Pi_V = \cup_a \Pi^a \tag{70}$$

denote the union of the vertex prisms and half-prisms respectively. From (52) and (69),

$$\begin{aligned}
\|\mathbf{w}\|^2_{1,2}(\Pi_V) \le 2\|\mathbf{w} - \tilde{\mathbf{n}}\|^2_{1,2}(\Pi_V) + 2\|\tilde{\mathbf{n}}\|^2_{1,2}(\Pi_V) \le& \\
\le \left(\left(5 + 8\textstyle\sum_b C^b\right)\epsilon_1 + 4\textstyle\sum_b C^b L^b {\epsilon_2}^2\right) :=& \ \epsilon_3.
\end{aligned} \tag{71}$$

Thus, the conditions of Lemma 3.1 are satisfied for each $\Lambda^a(H)$, and we can construct a smooth tangent unit-vector field $\mathbf{n}$ on Λ_V coinciding with $\mathbf{w}$ on $\Lambda_V \setminus (\cup_a \Lambda^a(\frac{3}{4}H)$ such that

$$\|\mathbf{n}\|^2_{1,2}(\Lambda_V) \le \textstyle\sum_a C^a \epsilon_3 \tag{72}$$

where $C^a > 0$ is independent of H, ϵ_3 and $\tilde{\mathbf{n}}$.

We extend $\mathbf{n}$ (smoothly) to all of P by taking $\mathbf{n}$ to coincide with $\mathbf{w}$ on $P \setminus \Lambda_V$. From (52), (69) and (72),

$$\begin{aligned}
\|\mathbf{n}-\tilde{\mathbf{n}}\|_{1,2}^2(P) &= \|\mathbf{n}-\tilde{\mathbf{n}}\|_{1,2}^2(P\setminus\Lambda_V) + \|\mathbf{n}-\tilde{\mathbf{n}}\|_{1,2}^2(\Lambda_V) \\
&\le \|\mathbf{w}-\tilde{\mathbf{n}}\|_{1,2}^2(P\setminus\Lambda_V) + 2\left(\|\mathbf{n}\|_{1,2}^2 + \|\tilde{\mathbf{n}}\|_{1,2}^2\right)(\Lambda_V) \\
&\le \left(\tfrac{1}{2} + 2{\textstyle\sum_a} C^a\right)\epsilon_3 + \epsilon_1. \qquad (73)
\end{aligned}$$

By choosing ϵ_1 and ϵ_2 appropriately, $\|\mathbf{n}-\tilde{\mathbf{n}}\|_{1,2}^2$ can be made arbitrarily small. From (61), $|\mathbf{n}-\tilde{\mathbf{n}}| \le \frac{1}{2}\epsilon_2$ on P'. This ensures that $\mathbf{n}$ and $\tilde{\mathbf{n}}$ have the same kink numbers and wrapping numbers. The construction also ensures that their edge orientations are the same. □

4 Discussion

We have derived a lower bound, depending on homotopy type, for the energies of harmonic maps of convex polyhedra in $\mathbb{R}^3$ to the unit sphere S^2 which satisfy tangent boundary conditions and are continuous everywhere except at vertices. It is natural to ask whether this lower bound is sharp. For the case of a rectangular prism, eg a cube, numerical calculations and analytical arguments have indicated that, for a large set of homotopy types, the lower bound (12) is not sharp, but differs from the actual lower bound by a fixed quantity independent of homotopy type [4]. A related question is whether the infimum (3) is achieved by some $\mathbf{n} \in C^0_{\mathfrak{T}}(P) \cap W^{1,2}(P)$. Numerics and analytic arguments for a rectangular prism have suggested that for a small number of "unwrapped" states, the infimum is achieved, but for a large set of homotopy types, a sequence of configurations with energies approaching $M(h)$ develop discontinuities along the edges. We plan to present these results in a subsequent paper.

While our present motivation comes from liquid crystal physics, related considerations for maps of two- (three-) dimensional polyhedra to spheres may be relevant for string (M-) theories with tangent boundary conditions ('T branes').

Acknowledgment

We are grateful to Adrian Geisow and Chris Newton, Hewlett-Packard Laboratories, Bristol for stimulating out interest in this area. AM was supported by an Industrial CASE studentship funded by EPSRC and by Hewlett-Packard Laboratories, Bristol.

References

[1] C Oseen, Theory of Liquid Crystals, *Trans. Faraday Soc* **29**, 883-899, 1933.
F.C. Frank, On the theory of liquid crystals, *Discuss. Faraday Soc.* **25**,19-28, 1958.
L.D. Landau and E.M. Lifshitz, *Theory of elasticity*, 3rd English ed., Pergamon Press, 1986.
P.G. de Gennes and J. Prost, *The physics of liquid crystals*, 2nd ed, Oxford, Clarendon Press, 1995.

[2] J.M. Robbins and M. Zyskin, Classification of unit-vector fields in convex polyhedra with tangent boundary conditions, Preprint math-ph/0402025.

[3] C.J.P. Newton and T.P. Spiller, Proc. 17th IDRC (SID), 1997.

[4] A. Majumdar, J.M. Robbins and M. Zyskin, in preparation.

[5] H. Brezis, J.M. Coron and E. Lieb, Harmonic maps with defects, *Comm. Math. Phys.* **107** 649-705, 1986.

[6] R.A. Adams, *Sobolev spaces*, Academic Press, 1975.